HW12

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6(a) Let $f(z) = -5z^4$, $g(z) = z^6 + z^3 - 2z$

On the circle $|z|=1$, $|g(z)| \leq 4 < 5 = |f(z)|$

So $f+g = z^6 - 5z^4 + z^3 - 2z$ has the same no. of zeros as f inside $|z|=1$, that is 4.

(b) Let $f(z) = 9$, $g(z) = 2z^4 - 2z^3 + 2z^2 - 2z$ and following the above arguement, we obtain that the no. of zeros of the polynomial is 0.

(c) Let $f(z) = -4z^3$ and $g(z) = z^7 + z - 1$. Similar, the no. of zeros is 3.

7(a) Let $f(z) = 9z^2$, $g(z) = z^4 - 2z^3 + z - 1$. As above, no. of zeros is 2.

(b) Let $f(z) = z^5$, $g(z) = 3z^3 + z^2 + 1$ As above, no. of zeros is 5

8. Let $f_1(z) = 2z^5$, $g_1(z) = -6z^2 + z + 1$

On $|z| = 2$, $|g_1(z)| \leq 6|z|^2 + |z| + 1 = 27 < 64 = |f_1(z)|$

So $f_1 + g_1$ has 5 zeros inside the circle $|z| = 2$.

Similarly, let $f_2(z) = -6z^2$, $g_2(z) = 2z^5 + z + 1$, we have $f_2 + g_2$ has 2 zeros inside the circle $|z| = 1$.

So $2z^5 - 6z^2 + z + 1$ has 3 zeros in the annulus $1 \leq |z| \leq 2$

9. Let $f(z) = cz^n$, $g(z) = -e^z$. On the circle $|z|=1$,

$|g(z)| \leq e^{\operatorname{Re} z} \leq e < |c| = |f(z)|$.

So $f+g = cz^n - e^z$ has the same no. of zeros as $f = cz^n$ inside $|z|=1$, that is n

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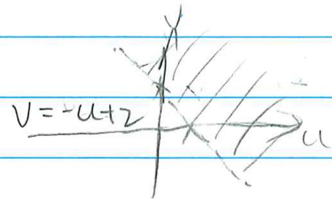
Let $z = x+iy$, $w = u+iv$.

2. $w = i(x+iy)+i = -y+(1+x)i$.

$v = 1+x > 1 \iff x > 0$. So the result follows.

5. $w = (1-i)(x+iy) = x+y+i(-x+y)$



$y > 1 \iff u+v = 2y > 2$

So the image of the half plane $y>1$ under the transformation $w=(1-i)z$ is

$v > -u+2$

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(for Q3,4) From example 22.3 of lecture notes, $y=c$, $c \neq 0$, is mapped by $w=\frac{1}{z}$ onto $\{w \neq 0 : |w + \frac{1}{2c}| = \frac{1}{2|c|}\}$.

3. $\boxed{\text{For } c_2 > 0}$, $\{z : \operatorname{Im} z > c_2\} = \bigcup_{c > c_2} \{z : \operatorname{Im} z = c\}$

and $\{w : |w + \frac{1}{2c_2}| < \frac{1}{2c_2}\} = \bigcup_{c > c_2} \{w \neq 0 : |w + \frac{1}{2c}| = \frac{1}{2c}\}$

So the half plane $y > c_2$ is mapped onto the interior of the circle $\{w : |w + \frac{1}{2c_2}| = \frac{1}{2c_2}\}$.

$\boxed{\text{Now let } c_2 = 0}$. Image of $\{y = 0\}$ is the real axis except 0.

$\{z : \operatorname{Im} z = 0\} = \bigcup_{c > 0} \{z : \operatorname{Im} z = c\}$. and

$\{w : \operatorname{Im} w < 0\} = \bigcup_{c > 0} \{w \neq 0 : |w + \frac{1}{2c}| = \frac{1}{2c}\}$

So the image is the lower half plane under the real axis.

$\boxed{\text{Let } c_2 < 0}$. Note that $w = \frac{1}{z}$ is a bijection from $\mathbb{C} \setminus \{0\}$ to itself.

Also, $\{z : \operatorname{Im} z < c_2\} = \bigcup_{c < c_2} \{z : \operatorname{Im} z = c\}$ and

$\{w \neq 0 : |w + \frac{1}{2c_2}| \leq \frac{1}{2|c_2|}\} = \bigcup_{c < c_2} \{w \neq 0 : |w + \frac{1}{2c}| = \frac{1}{2|c|}\}$

So the image is $\mathbb{C} \setminus \{w : |w + \frac{1}{2c_2}| \leq \frac{1}{2|c_2|}\}$

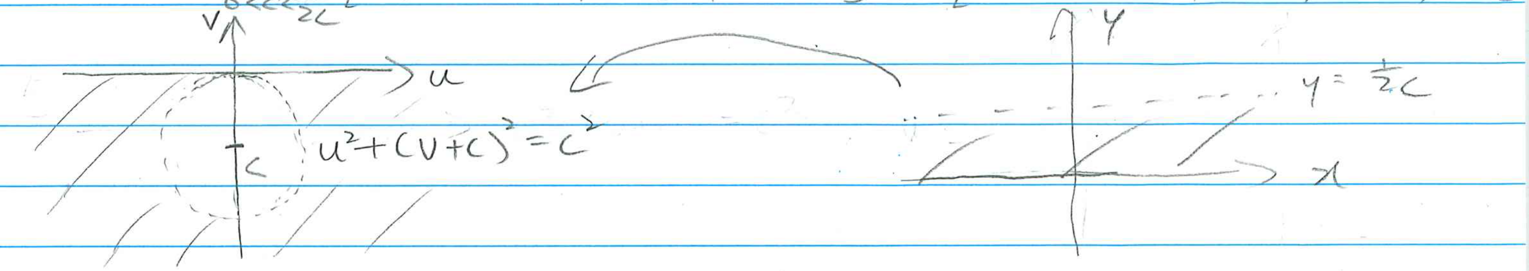
4. As before, $\{z : 0 < \text{Im } z < \frac{1}{2c}\} = \bigcup_{0<c'<c} \{z : \text{Im } z = c'\}$

and the image of each set $\{z : \text{Im } z = c'\}$ under the mapping $w = \frac{1}{z}$,

$\{w \neq 0 : |w + \frac{1}{2c'}| = \frac{1}{2c}\}$.

So the image of the strip $0 < y < \frac{1}{2c}$ is

$$\bigcup_{0<c'<\frac{1}{2c}} \{w \neq 0 : |w + \tfrac{1}{2c'}| = \tfrac{1}{2c'}\} = \{w = u + iv : u^2 + (v+c)^2 > c^2, v < 0\}$$



5. From example 22.4 of lecture notes,

$\{z : \text{Re } z > 1\}$ is mapped onto $\{w : |w - \frac{1}{2}| < \frac{1}{2}\}$

Note that $w = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$.

We have $y > 0 \iff v < 0$.

So the image of the region $x > 1$, $y > 0$ under the transformation is $(u - \frac{1}{2})^2 + v^2 < (\frac{1}{2})^2$, $v < 0$.